LOCATION OF BLOW UP POINTS OF LEAST ENERGY SOLUTIONS TO THE BREZIS-NIRENBERG EQUATION (Variational Problems and Related Topics)

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LOCATION OF BLOW UP POINTS OF LEAST ENERGY SOLUTIONS TO THE BREZIS-NIRENBERG EQUATION

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1. Introduction. Let Ω be a smooth bounded domain in $\mathbf{R}^N, N \geq 4$ and $p = \frac{N+2}{N-2}$. In this article, we return to the well-studied problem (P_ε):

$$\begin{cases} -\Delta u = u^p + \varepsilon u & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0, \end{cases}$$

where $\varepsilon > 0$ is a parameter.

The exponent p is called the critical Sobolev exponent in the sense that the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{p+1}(\Omega)$ is continuous but not compact. So from the variational view point, this problem belongs to the limit case of the Palais-Smale compactness condition, and the classical arguments do not apply to the questions related to the existence or nonexistence and multiplicity of solutions of this problem.

In pioneering work [3], Brezis and Nirenberg proved that, in spite of possible failure of the Palais- Smale compactness condition, (P_ε) has at least one non-trivial solution on a general bounded domain Ω when $\varepsilon \in (0, \lambda_1)$, where λ_1 denotes the first eigenvalue of $-\Delta$ with Dirichlet boundary condition.

On the other hand when $\varepsilon = 0$, it is known that problem (P_0) reflects the topology and the geometry of the domain Ω. Pohozaev showed that if Ω is star-shaped, then (P_0) has no non-trivial solutions [7]. In other cases Bahri and Coron [1] proved that (P_0) has a solution when Ω has non-trivial topology in the sense that $H_d(\Omega, \mathbf{Z}_2) \neq \{0\}$ for some positive integer d, where $H_d(\Omega, \mathbf{Z}_2)$ denotes the d-th homology group of Ω with $\mathbf{Z}_2$ coefficients. Furthermore Ding [5] and Passaseo [8] proved that even if Ω is contractible, (P_0) can still have a solution if the geometry of Ω is non-trivial in some sense.

Because of the different nature of the problem when $\varepsilon > 0$ and $\varepsilon = 0$, it is interesting to study the asymptotic behavior of solutions u_ε of (P_ε) as $\varepsilon \to 0$. In this direction, Han [9] and Rey [12][13] proved independently the following result, which had been conjectured previously by Brezis and Peletier [4].

Theorem 0.(Han [9], Rey [12]) *Let u_ε be a solution of problem (P_ε) and assume*

$$\frac{\int_\Omega |\nabla u_\varepsilon|^2 dx}{(\int_\Omega |u_\varepsilon|^{p+1} dx)^{\frac{2}{p+1}}} = S + o(1) \quad \text{as } \varepsilon \to 0,$$

where S is the best Sobolev constant in $\mathbf{R}^N$:

$$S = \pi N(N-2)\left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)}\right)^{\frac{2}{N}}.$$

Then we have (after passing to a subsequence):

(1) There exists $a_\infty \in \Omega$ (interior point) such that

$$|\nabla u_\varepsilon|^2 \overset{*}{\rightharpoonup} S^{\frac{N}{2}} \delta_{a_\infty} \quad \text{as } \varepsilon \to 0$$

in the sense of Radon measures of the compact space $\overline{\Omega}$, where δ_a is the Dirac measure supported by $a \in \mathbf{R}^N$.

(2) The a_∞ above is a critical point of the (positive) Robin function $H(a,a)$ on Ω:

$$\nabla_a H(a_\infty, a_\infty) = 0,$$

where $H(x,a)$ is the regular part of the Green's function $G(x,a)$:

$$H(x,a) := \frac{1}{(N-2)\omega_N}|x-a|^{2-N} - G(x,a),$$

in which $\omega_N = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the $(N-1)$ dimensional volume of S^{N-1} and

$$\begin{cases} -\Delta_x G(x,a) = \delta_a(x), \ x \in \Omega, \\ G(x,a)|_{x\in\partial\Omega} = 0. \end{cases}$$

(3) We have an exact blow up rate of the L^∞-norm of u_ε as $\varepsilon \to 0$:

$$\lim_{\varepsilon\to 0} \varepsilon \|u_\varepsilon\|_{L^\infty(\Omega)}^{\frac{2(N-4)}{N-2}} = (N(N-2))^{\frac{N-4}{2}} \frac{(N-2)^3\omega_N}{2C_N} H(a_\infty, a_\infty), \quad \text{if } N \geq 5,$$

$$\lim_{\varepsilon\to 0} \varepsilon \log \|u_\varepsilon\|_{L^\infty(\Omega)} = 4\omega_4 H(a_\infty, a_\infty), \quad \text{if } N = 4,$$

where

$$C_N = \int_0^\infty \frac{s^{N-1}}{(1+s^2)^{N-2}} ds = \frac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{2\Gamma(N-2)}.$$

In this article, we restrict our attention to a particular family of solutions to (P_ε), namely the solutions $(\overline{u}_\varepsilon)_{\varepsilon\in(0,\lambda_1)}$ obtained by the method of Brezis and Nirenberg. We call $(\overline{u}_\varepsilon)$ the *least energy solutions* to the problem (P_ε).

Before stating our main result, we recall the construction of least energy solutions by Brezis and Nirenberg.

For $\varepsilon \in (0, \lambda_1)$, define

$$S_\varepsilon := \inf_{\substack{u\in H_0^1(\Omega) \\ \|u\|_{L^{p+1}(\Omega)}=1}} \left\{ \int_\Omega |\nabla u|^2 dx - \varepsilon \int_\Omega u^2 dx \right\}. \tag{1.1}$$

Since the constraint on $\|u\|_{L^{p+1}(\Omega)}$ is not preserved under weak convergence in $H_0^1(\Omega)$, it is not obvious that S_ε is achieved or not. By using the fact that $S_\varepsilon < S$ if $\varepsilon > 0$, Brezis-Nirenberg proved that any minimizing sequence for (1.1) is compact in $H_0^1(\Omega)$ and (1.1) is achieved by some positive function $v_\varepsilon^0 \in H_0^1(\Omega)$. Furthermore if $\varepsilon < \lambda_1$, then it follows $S_\varepsilon > 0$ and

$$\overline{u}_\varepsilon := S_\varepsilon^{\frac{N-2}{4}} v_\varepsilon^0 \tag{1.2}$$

is a solution to (P_ε).

By Global Compactness Theorem of Struwe [14], we know that the least energy solutions $\overline{u}_\varepsilon$ blow up at exactly one point in $\overline{\Omega}$ as $\varepsilon \to 0$. That is, there exist $\lambda_\varepsilon > 0$ with $\lambda_\varepsilon \to 0$ $(\varepsilon \to 0)$ and $a_\varepsilon \in \Omega$ with $\lambda_\varepsilon/\mathrm{dist}(a_\varepsilon, \partial\Omega) \to 0$ $(\varepsilon \to 0)$ such that

$$\|\nabla(\overline{u}_\varepsilon - \alpha_N PU_{\lambda_\varepsilon,a_\varepsilon})\|_{L^2(\Omega)} \to 0 \quad \text{as } \varepsilon \to 0, \tag{1.3}$$

where $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$.

Here for $\lambda > 0$ and $a \in \Omega$, we define

$$U_{\lambda,a}(x) := \left(\frac{\lambda}{\lambda^2 + |x-a|^2} \right)^{\frac{N-2}{2}}, \quad x \in \mathbf{R}^N \tag{1.4}$$

and $PU_{\lambda,a} := U_{\lambda,a} - \varphi_{\lambda,a} \in H_0^1(\Omega)$, where $\varphi_{\lambda,a}$ is the harmonic extension of $U_{\lambda,a}|_{\partial\Omega}$ to Ω:

$$\begin{cases} -\Delta\varphi_{\lambda,a} = 0 \text{ in } \Omega, \\ \varphi_{\lambda,a}|_{\partial\Omega} = U_{\lambda,a}|_{\partial\Omega}. \end{cases} \tag{1.5}$$

We call any accumulation point of $(a_\varepsilon)_{\varepsilon>0}$ *a blow up point* of $(\overline{u}_\varepsilon)$. Note that if $a_\infty \in \overline{\Omega}$ is a blow up point of $(\overline{u}_\varepsilon)_{\varepsilon>0}$, then by passing to a subsequence, we see $|\nabla \overline{u}_\varepsilon|^2 \overset{*}{\rightharpoonup} S^{\frac{N}{2}}\delta_{a_\infty}$ as $\varepsilon \to 0$, and by construction, $(\overline{u}_\varepsilon)$ is a minimizing sequence for the best Sobolev constant. So from the result of Han and Rey, we know that $a_\infty \in \Omega$ (interior point) and a_∞ is a critical point of the Robin function on Ω.

Our main result is to further locate the blow up point a_∞ of the least energy solutions on a general bounded domain Ω in $\mathbf{R}^N$, $N \geq 4$.

Theorem 1. *Let a_∞ be a blow up point of the least energy solutions $(\overline{u}_\varepsilon)$ obtained by the method of Brezis and Nirenberg. Then a_∞ is a minimum point of the Robin function of Ω:*

$$H(a_\infty, a_\infty) = \inf_{a\in\Omega} H(a, a).$$

To prove Theorem 1, we will make a precise asymptotic expansion of the value S_ε as $\varepsilon \to 0$. For this purpose, we combine the method developed by Isobe [10] [11] and technical calculations in Rey [12] [13]. As a by-product of our method, we prove that the blow up point is the interior point of Ω by using only an energy comparison argument. Also we can give another explanation of the exact blow up rate of L^∞-norm of $\overline{u}_\varepsilon$ along the line of our context.

Wei [15] treated the subcritical problem:

$$\begin{cases} -\Delta u = u^{p-\varepsilon} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u|_{\partial\Omega} = 0 \end{cases}$$

where $\varepsilon > 0$, and he proved that as $\varepsilon \to 0$, the least energy solutions to this problem blow up at exactly one point, and the blow up point is a minimum point of the Robin function. His method is the usual blow-up (rescaling) technique and he obtained a second order expansion of the rescaled function, which leads to an asymptotic expansion as $\varepsilon \to 0$ of the value

$$\inf_{\substack{u\in H_0^1(\Omega) \\ \|u\|_{L^{p+1-\varepsilon}(\Omega)}=1}} \left\{ \int_\Omega |\nabla u|^2 dx \right\}.$$

In the course of the proof, he used the result of Han and Rey, and a crucial pointwise estimate obtained by Han for the rescaled function.

We might follow the method of Wei to study the problem (P_ε) when $N \geq 5$, but even in this case, I believe that our method is more consistent and somewhat simpler because we do not need any use of Pohozaev identity, Kelvin transformation and Gidas-Ni-Nirenberg theory. See also [6].

2. Asymptotic behavior of S_ε. In this section, we obtain an asymptotic formula of the value S_ε as $\varepsilon \to 0$ and derive the suitable upper bound for S_ε. See Lemma 2.5 and Lemma 2.7.

For $\varepsilon \in (0, \lambda_1)$, let $v_\varepsilon^0 \in H_0^1(\Omega)$ be a solution to the minimization problem (1.1).

Define

$$v_\varepsilon := S^{\frac{N-2}{4}} v_\varepsilon^0. \tag{2.1}$$

Then (1.2), (1.3) and $S_\varepsilon = S + o(1)$ as $\varepsilon \to 0$ imply

$$\begin{aligned} \|\nabla(v_\varepsilon - \alpha_N PU_{\lambda_\varepsilon, a_\varepsilon})\|_{L^2(\Omega)} &\to 0 \quad \text{as } \varepsilon \to 0, && (2.2) \\ \int_\Omega v_\varepsilon^{p+1} dx &= S^{\frac{N}{2}}. && (2.3) \end{aligned}$$

Define for $\eta > 0$,

$$M(\eta) := \left\{ v \in H_0^1(\Omega) : \begin{array}{l} \exists \alpha > 0, |\alpha - \alpha_N| < \eta, \exists a \in \Omega, \exists \lambda > 0 \\ \text{with } \lambda / d(a, \partial\Omega) < \eta \\ \text{such that } \|\nabla(v - \alpha PU_{\lambda,a})\|_{L^2(\Omega)} < \eta. \end{array} \right\}$$

where $d(a, \partial\Omega) = \mathrm{dist}(a, \partial\Omega)$.

It is proved in [1]:Proposition 7, that for $v \in M(\eta)$ and $\eta > 0$ small enough, the minimization problem:

$$\text{Minimize} \left\{ \|\nabla(v - \alpha PU_{\lambda,a})\|_{L^2(\Omega)} : \begin{array}{l} \alpha \in (\alpha_N - 2\eta, \alpha_N + 2\eta), \\ \lambda > 0, a \in \Omega, \\ \lambda / d(a, \partial\Omega) < 2\eta \end{array} \right\} \tag{2.4}$$

has a unique solution $(\alpha^0, \lambda^0, a^0) \in (\alpha_N - 2\eta, \alpha_N + 2\eta) \times \mathbf{R}_+ \times \Omega$.

Let $a_\infty \in \overline{\Omega}$ be a blow up point of $(\overline{u}_\varepsilon)_{\varepsilon>0}$. By definition of the blow up point, there exist $\varepsilon_n \to 0, \lambda_n \to 0, \Omega \ni a_n \to a_\infty$ such that $(v_n := v_{\varepsilon_n}, d_n := \mathrm{dist}(a_n, \partial\Omega))$

$$\|\nabla(v_n - \alpha_N PU_{\lambda_n, a_n})\|_{L^2(\Omega)} \to 0, \quad \lambda_n / d_n \to 0 \; (n \to \infty). \tag{2.5}$$

(2.5) implies there exists $\eta_n \to 0$ such that $v_n \in M(\eta_n)$. We denote the unique solution $(\alpha_n^0, \lambda_n^0, a_n^0)$ to (2.4) for $v = v_n, \eta = \eta_n$ again by $(\alpha_n, \lambda_n, a_n)$.

Then by our choice of $(\alpha_n, \lambda_n, a_n)$, if we write

$$v_n = \alpha_n PU_{\lambda_n, a_n} + w_n, \quad w_n \in H_0^1(\Omega), \tag{2.6}$$

it follows that

$$\begin{aligned} \alpha_n &\to \alpha_N = (N(N-2))^{\frac{N-2}{4}},\ a_n \to a_\infty, \\ \frac{\lambda_n}{d_n} &\to 0 \quad \text{where } d_n = \operatorname{dist}(a_n, \partial\Omega), \\ w_n &\in E_{\lambda_n, a_n},\ w_n \to 0 \text{ in } H_0^1(\Omega) \end{aligned} \tag{2.7}$$

as $n \to \infty$. Here for $\lambda > 0$ and $a \in \Omega$,

$$\begin{aligned} E_{\lambda,a} := \{w \in H_0^1(\Omega) :\ 0 = &\int_\Omega \nabla w \cdot \nabla PU_{\lambda,a} dx \\ = &\int_\Omega \nabla w \cdot \nabla(\frac{\partial}{\partial a_i} PU_{\lambda,a}) dx \quad (i = 1, \cdots, N) \\ = &\int_\Omega \nabla w \cdot \nabla(\frac{\partial}{\partial \lambda} PU_{\lambda,a}) dx\}. \end{aligned} \tag{2.8}$$

In the following, we estimate

$$J_n := \int_\Omega |\nabla v_n|^2 dx - \varepsilon_n \int_\Omega v_n^2 dx \tag{2.9}$$

by using the expression (2.6).

Lemma 2.1.(Asymptotic behavior of H_0^1 norm of the main part) *As $n \to \infty$, we have*

$$\begin{aligned} \int_\Omega |\nabla PU_{\lambda_n, a_n}|^2 dx &= N(N-2)A - (N-2)^2 \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} \\ &+ O\left(\frac{\lambda_n^N}{d_n^N} |\log(\frac{\lambda_n}{d_n})| \right), \end{aligned}$$

where

$$A = \int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p+1} dx = \frac{\Gamma(N/2)}{\Gamma(N)} \pi^{N/2}.$$

Proof. We have

$$\begin{aligned} &\int_\Omega |\nabla PU_{\lambda_n,a_n}|^2 dx = \int_\Omega -\Delta PU_{\lambda_n,a_n} \cdot PU_{\lambda_n,a_n} dx \\ =\ & N(N-2) \int_\Omega U_{\lambda_n,a_n}^p \cdot (U_{\lambda_n,a_n} - \varphi_{\lambda_n,a_n}) dx \\ =\ & N(N-2) \int_\Omega U_{\lambda_n,a_n}^{p+1} dx - N(N-2) \int_\Omega U_{\lambda_n,a_n}^p \varphi_{\lambda_n,a_n} dx \\ =:\ & N(N-2) I_1 - N(N-2) I_2. \end{aligned} \tag{2.10}$$

Here we have used the fact that $PU_{\lambda_n,a_n} \in H_0^1(\Omega)$ satisfies the equation

$$-\Delta PU_{\lambda_n,a_n} = N(N-2)U^p_{\lambda_n,a_n} \quad \text{in } \Omega. \tag{2.11}$$

Now,

$$\begin{aligned}
I_1 &= \int_\Omega U^{p+1}_{\lambda_n,a_n}dx = \int_{\mathbf{R}^N} U^{p+1}_{\lambda_n,a_n}dx - \int_{\mathbf{R}^N\setminus\Omega} U^{p+1}_{\lambda_n,a_n}dx \\
&= A + O\left(\int_{\mathbf{R}^N\setminus B_{d_n}(a_n)} U^{p+1}_{\lambda_n,a_n}dx\right) \\
&= A + O\left(\lambda_n^N \int_{r=d_n}^{r=\infty} \frac{r^{N-1}}{(\lambda_n^2+r^2)^N}dr\right) \qquad (r=|x-a_n|) \\
&= A + O\left(\frac{\lambda_n^N}{d_n^N}\right).
\end{aligned} \tag{2.12}$$

We divide I_2 in the second term of (2.10) as

$$\begin{aligned}
I_2 &= \int_\Omega U^p_{\lambda_n,a_n}\varphi_{\lambda_n,a_n}dx \\
&= \int_{\Omega\setminus B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n}\varphi_{\lambda_n,a_n}dx + \int_{B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n}\varphi_{\lambda_n,a_n}dx \\
&=: I_2^1 + I_2^2.
\end{aligned} \tag{2.13}$$

Now,

$$\begin{aligned}
I_2^1 &= \int_{\Omega\setminus B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n}\varphi_{\lambda_n,a_n}dx \\
&= O\left(\|\varphi_{\lambda_n,a_n}\|_{L^\infty(\Omega)} \int_{\Omega\setminus B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n}dx\right) \\
&= O\left((\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^{N-2}})\cdot \lambda_n^{\frac{N+2}{2}} \int_{r=d_n/2}^{r=\infty} \frac{r^{N-1}}{(\lambda_n^2+r^2)^{\frac{N+2}{2}}}dr\right) \\
&= O\left(\frac{\lambda_n^N}{d_n^N}\right).
\end{aligned} \tag{2.14}$$

Here, we have used the estimate

$$\|\varphi_{\lambda_n,a_n}\|_{L^\infty(\Omega)} = O\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^{N-2}}\right), \tag{2.15}$$

which is a consequence of (1.5) and the maximum principle of harmonic functions.

In calculating I_2^2, we make a Taylor expansion of φ_{λ_n,a_n} on $B_{d_n/2}(a_n)$:

$$\begin{aligned}\varphi_{\lambda_n,a_n} = \varphi_{\lambda_n,a_n}(a_n) &+ \nabla\varphi_{\lambda_n,a_n}(a_n)\cdot(x-a_n)\\ &+ O\left(\|\nabla^2\varphi_{\lambda_n,a_n}\|_{L^\infty(B_{d_n/2}(a_n))}|x-a_n|^2\right).\end{aligned}$$

Note that we have

$$\varphi_{\lambda_n,a_n}(a_n) = (N-2)\omega_N\lambda_n^{\frac{N-2}{2}}H(a_n,a_n) + O\left(\frac{\lambda_n^{\frac{N+2}{2}}}{d_n^N}\right) \tag{2.16}$$

by [13]:Proposition 1, and

$$\|\nabla^2\varphi_{\lambda_n,a_n}\|_{L^\infty(B_{d_n/2}(a_n))} = O\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^N}\right) \tag{2.17}$$

by the elliptic estimate $d_n^k\|\nabla^k\varphi_{\lambda_n,a_n}\|_{L^\infty(B_{d_n/2}(a_n))} \le \|\varphi_{\lambda_n,a_n}\|_{L^\infty(\Omega)}$ $(k\in\mathbf{N})$ for a harmonic function φ_{λ_n,a_n}.

Then by (2.16), (2.17) and the oddness of the integral, we calculate:

$$\begin{aligned} I_2^2 &= \int_{B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n}\varphi_{\lambda_n,a_n}dx\\ &= \int_{B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n}\varphi_{\lambda_n,a_n}(a_n)dx\\ &+ \int_{B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n}\nabla\varphi_{\lambda_n,a_n}(a_n)\cdot(x-a_n)dx\\ &+ \int_{B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n}\cdot O(\|\nabla^2\varphi_{\lambda_n,a_n}\|_{L^\infty(B_{d_n/2}(a_n))}|x-a_n|^2)dx\\ &= \{(N-2)\omega_N\lambda_n^{\frac{N-2}{2}}H(a_n,a_n) + O(\frac{\lambda_n^{\frac{N+2}{2}}}{d_n^N})\}\int_{B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n}dx + 0\\ &+ O\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^N}\int_{B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n}|x-a_n|^2dx\right)\\ &= (\frac{N-2}{N})\omega_N^2\lambda_n^{N-2}H(a_n,a_n) + O(\frac{\lambda_n^N}{d_n^N}) + O\left(\frac{\lambda_n^N}{d_n^N}|\log(\frac{\lambda_n}{d_n})|\right). \end{aligned} \tag{2.18}$$

Here in the last equality, we have used the estimates

$$\int_{B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n}dx = \omega_N\int_0^{d_n/2}\left(\frac{\lambda_n}{\lambda_n^2+r^2}\right)^{\frac{N+2}{2}} r^{N-1}dr$$

$$
\begin{aligned}
&= \omega_N \lambda_n^{\frac{N-2}{2}} \int_0^{d_n/2\lambda_n} \frac{s^{N-1}}{(1+s^2)^{\frac{N+2}{2}}} ds = \omega_N \lambda_n^{\frac{N-2}{2}} \left(\int_0^{\infty} - \int_{d_n/2\lambda_n}^{\infty} \right) \\
&= \frac{\omega_N}{N} \lambda_n^{\frac{N-2}{2}} + O\left(\frac{\lambda_n^{\frac{N+2}{2}}}{d_n^2} \right), \qquad (2.19)
\end{aligned}
$$

$$
\begin{aligned}
&\int_{B_{d_n/2}(a_n)} U^p_{\lambda_n,a_n} O(|x-a_n|^2) dx = O\left(\lambda_n^{\frac{N+2}{2}} \int_0^{d_n/2\lambda_n} \frac{s^{N+1}}{(1+s^2)^{\frac{N+2}{2}}} ds \right) \\
&= O\left(\lambda_n^{\frac{N+2}{2}} |\log(\frac{\lambda_n}{d_n})| \right), \qquad (2.20)
\end{aligned}
$$

and the estimate of the Robin function:

$$
H(a_n, a_n) = \frac{1}{(N-2)\omega_N} (\frac{1}{2d_n})^{N-2} + o(\frac{1}{d_n^{N-2}}) \quad \text{as } d_n \to 0 \qquad (2.21)
$$

(see [13]:(2.8)).

(2.19) is a consequence of

$$
\int_0^{\infty} \frac{s^{N-1}}{(1+s^2)^{\frac{N+2}{2}}} ds = \frac{\Gamma(\frac{N}{2})\Gamma(1)}{2\Gamma(\frac{N+2}{2})} = \frac{1}{N},
$$

where we used the formula

$$
\int_0^{\infty} \frac{s^{\alpha}}{(1+s^2)^{\beta}} ds = \frac{\Gamma(\frac{1+\alpha}{2})\Gamma(\frac{2\beta-\alpha-1}{2})}{2\Gamma(\beta)} \qquad (2.22)
$$

for $\alpha > 0, \beta > 0$ and $2\beta - \alpha - 1 > 0$.

From (2.10)-(2.18), we obtain the conclusion of Lemma 2.1. □

Lemma 2.2.(Asymptotic behavior of L^2 norm of the main part) *When $N \geq 5$, we have*

$$
\int_{\Omega} PU^2_{\lambda_n,a_n} dx = \omega_N C_N \lambda_n^2 + o(\lambda_n^2) \quad \text{as } n \to \infty,
$$

where

$$
C_N = \int_0^{\infty} \frac{s^{N-1}}{(1+s^2)^{N-2}} ds = \frac{\Gamma(\frac{N}{2})\Gamma(\frac{N-4}{2})}{2\Gamma(N-2)}.
$$

When $N = 4$, we have

$$
\begin{aligned}
\int_{\Omega} PU^2_{\lambda_n,a_n} dx &= \omega_4 \lambda_n^2 |\log \lambda_n| + o(\lambda_n^2 |\log \lambda_n|) \\
&+ O\left(\frac{\lambda_n^2}{d_n} |\log \lambda_n|^{1/2} \right) + O\left(\frac{\lambda_n^2}{d_n^2} \right) \quad \text{as } n \to \infty,
\end{aligned}
$$

Proof ($N \geq 5$). We extend PU_{λ_n,a_n} and φ_{λ_n,a_n} to $\mathbf{R}^N$ by setting $PU_{\lambda_n,a_n} = 0$ in $\mathbf{R}^N \setminus \Omega$ and $\varphi_{\lambda_n,a_n} = U_{\lambda_n,a_n}$ in $\mathbf{R}^N \setminus \Omega$. We denote them again by PU_{λ_n,a_n} and φ_{λ_n,a_n} respectively.

Since $PU_{\lambda_n,a_n} = U_{\lambda_n,a_n} - \varphi_{\lambda_n,a_n}$, we have

$$\begin{aligned}\int_\Omega PU^2_{\lambda_n,a_n}dx &= \int_\Omega U^2_{\lambda_n,a_n}dx + \int_\Omega \varphi^2_{\lambda_n,a_n}dx \\ &+ O\Big(\big(\int_\Omega U^2_{\lambda_n,a_n}dx\big)^{1/2}\big(\int_\Omega \varphi^2_{\lambda_n,a_n}dx\big)^{1/2}\Big). \qquad (2.23)\end{aligned}$$

We estimate the first term in (2.23) as follows: By monotonicity of the integral, we have

$$\int_{B_{d_n}(a_n)} U^2_{\lambda_n,a_n}dx \leq \int_\Omega U^2_{\lambda_n,a_n}dx \leq \int_{B_R(a_n)} U^2_{\lambda_n,a_n}dx, \qquad (2.24)$$

where $R = \mathrm{diam}(\Omega)$.

Calculation shows

$$\begin{aligned}&\int_{B_{d_n}(a_n)} U^2_{\lambda_n,a_n}dx = \omega_N \int_0^{d_n} \left(\frac{\lambda_n}{\lambda_n^2 + r^2}\right)^{N-2} r^{N-1}dr \\ &= \omega_N \lambda_n^2 \int_0^{d_n/\lambda_n} \frac{s^{N-1}}{(1+s^2)^{N-2}}ds \\ &= \omega_N \lambda_n^2 \left(\int_0^\infty - \int_{d_n/\lambda_n}^\infty\right) \\ &= \omega_N \lambda_n^2 \left(C_N + O\Big(\Big|\int_{d_n/\lambda_n}^\infty \frac{s^{N-1}}{(1+s^2)^{N-2}}ds\Big|\Big)\right) \\ &= \omega_N C_N \lambda_n^2 + O\left(\frac{\lambda_n^{N-2}}{d_n^{N-4}}\right),\end{aligned}$$

here we have used the assumption $N \geq 5$.

The same calculation shows

$$\int_{B_R(a_n)} U^2_{\lambda_n,a_n}dx = \omega_N C_N \lambda_n^2 + O(\lambda_n^{N-2}).$$

So dividing both the integrals of (2.24) by $\omega_N C_N \lambda_n^2$ and noting $(\lambda_n/d_n) = o(1)$ (see (2.7)), we obtain

$$\lim_{n\to\infty} \frac{\int_\Omega U^2_{\lambda_n,a_n}dx}{\omega_N C_N \lambda_n^2} = 1,$$

$$\int_\Omega U^2_{\lambda_n,a_n}dx = \omega_N C_N \lambda_n^2 + o(\lambda_n^2) \quad (n \to \infty). \tag{2.25}$$

To estimate the second term in (2.23), we divide the integral in two parts:

$$\int_\Omega \varphi^2_{\lambda_n,a_n}dx = \int_{B_{d_n}(a_n)} \varphi^2_{\lambda_n,a_n}dx + \int_{\Omega\setminus B_{d_n}(a_n)} \varphi^2_{\lambda_n,a_n}dx.$$

Then:

$$\begin{aligned}\int_{B_{d_n}(a_n)} \varphi^2_{\lambda_n,a_n}dx &= O\left(\|\varphi_{\lambda_n,a_n}\|^2_{L^\infty(\Omega)} \cdot \mathrm{vol}(B_{d_n}(a_n))\right)\\ &= O\left(\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^{N-2}}\right)^2 \cdot d_n^N\right) = O\left(\frac{\lambda_n^{N-2}}{d_n^{N-4}}\right)\end{aligned}$$

by (2.15), and

$$\begin{aligned}\int_{\Omega\setminus B_{d_n}(a_n)} \varphi^2_{\lambda_n,a_n}dx &= O\left(\int_{\mathbf{R}^N\setminus B_{d_n}(a_n)} U^2_{\lambda_n,a_n}dx\right)\\ &= O\left(\int_{d_n}^\infty \left(\frac{\lambda_n}{\lambda_n^2 + r^2}\right)^{N-2} r^{N-1}dr\right)\\ &= O\left(\frac{\lambda_n^{N-2}}{d_n^{N-4}}\right),\end{aligned}$$

since $0 < \varphi_{\lambda_n,a_n} < U_{\lambda_n,a_n}$ in Ω and $\varphi_{\lambda_n,a_n} = U_{\lambda_n,a_n}$ on $\mathbf{R}^N \setminus \Omega$.

In conclusion, we have

$$\int_\Omega \varphi^2_{\lambda_n,a_n}dx = O\left(\frac{\lambda_n^{N-2}}{d_n^{N-4}}\right) = o(\lambda_n^2) \quad \text{as } n \to \infty. \tag{2.26}$$

By (2.23),(2.25) and (2.26), we have the conclusion of Lemma 2.2. □

From Lemma 2.1, Lemma 2.2 and the fact that

$$\int_\Omega |\nabla v_n|^2 dx = \alpha_n^2 \int_\Omega |\nabla PU_{\lambda_n,a_n}|^2 dx + \int_\Omega |\nabla w_n|^2 dx$$

(which follows since $w_n \in E_{\lambda_n,a_n}$; see (2.8)), we have the following lemma, for example when $N \geq 5$.

Lemma 2.3.(Asymptotic behavior of J_n) *When $N \geq 5$, we have*

$$\begin{aligned} J_n &:= \int_\Omega |\nabla v_n|^2 dx - \varepsilon_n \int_\Omega v_n^2 dx \\ &= \alpha_n^2 \left\{ N(N-2)A - (N-2)^2 \omega_N^2 H(a_n, a_n) \lambda_n^{N-2} \right\} - \varepsilon_n \alpha_n^2 \omega_N C_N \lambda_n^2 \\ &+ \|\nabla w_n\|_{L^2(\Omega)}^2 - \varepsilon_n \|w_n\|_{L^2(\Omega)}^2 + O\left(\frac{\lambda_n^N}{d_n^N} |\log(\frac{\lambda_n}{d_n})| \right) + o(\varepsilon_n \lambda_n^2) \\ &+ O(\varepsilon_n \lambda_n \|w_n\|_{L^2(\Omega)}) \quad \text{as } n \to \infty. \end{aligned}$$

To proceed further, we need the precise asymptotic behavior of α_n as $n \to \infty$. This is given by the next lemma.

Lemma 2.4.(Asymptotic behavior of α_n)
When $N \geq 4$, we have

$$\alpha_n^2 = \alpha_N^2 + \alpha_N^2 \left(\frac{N-2}{N} \right) \left(\frac{2\omega_N^2}{A} \right) H(a_n, a_n) \lambda_n^{N-2} + O\left(\|\nabla w_n\|_{L^2(\Omega)}^2 \right) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}} \right)$$

as $n \to \infty$, where $\alpha_N = (N(N-2))^{\frac{N-2}{4}}$.

Proof. After extending v_n, PU_{λ_n, a_n}, and w_n by 0 outside Ω, we have

$$S^{N/2} = \int_\Omega v_n^{p+1} dx = \int_{\mathbf{R}^N} |\alpha_n PU_{\lambda_n, a_n} + w_n|^{p+1} dx \tag{2.27}$$

by (2.3). We set $W_n := -\alpha_n \varphi_{\lambda_n, a_n} + w_n$, here as before, φ_{λ_n, a_n} is extended to $\mathbf{R}^N$ by U_{λ_n, a_n} on $\mathbf{R}^N \setminus \Omega$.

By expanding the right hand side of (2.27), we have

$$\begin{aligned} S^{N/2} &= \int_{\mathbf{R}^N} (\alpha_n U_{\lambda_n, a_n} + W_n)^{p+1} dx \\ &= \alpha_n^{p+1} \int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p+1} dx + (p+1) \alpha_n^p \int_{\mathbf{R}^N} U_{\lambda_n, a_n}^p W_n dx \\ &+ O\left(\int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p-1} W_n^2 dx + \int_{\mathbf{R}^N} |W_n|^{p+1} dx \right). \end{aligned} \tag{2.28}$$

First, we know

$$\alpha_n^{p+1} \int_{\mathbf{R}^N} U_{\lambda_n, a_n}^{p+1} dx = \alpha_n^{p+1} A. \tag{2.29}$$

Next, by using the equation $-\Delta U_{\lambda_n,a_n} = N(N-2)U^p_{\lambda_n,a_n}$ in $\mathbf{R}^N$, we calculate

$$
\begin{aligned}
&(p+1)\alpha_n^p \int_{\mathbf{R}^N} U^p_{\lambda_n,a_n} W_n dx = \frac{2\alpha_n^p}{(N-2)^2}\int_{\mathbf{R}^N}(-\Delta U_{\lambda_n,a_n})W_n dx \\
= \;& \frac{2\alpha_n^p}{(N-2)^2}\int_{\mathbf{R}^N}\nabla U_{\lambda_n,a_n}\cdot\nabla W_n dx \\
= \;& \frac{2\alpha_n^p}{(N-2)^2}\int_{\mathbf{R}^N}(\nabla PU_{\lambda_n,a_n}+\nabla\varphi_{\lambda_n,a_n})\cdot(-\alpha_n\nabla\varphi_{\lambda_n,a_n}+\nabla w_n)dx \\
= \;& \frac{-2\alpha_n^{p+1}}{(N-2)^2}\int_{\mathbf{R}^N}|\nabla\varphi_{\lambda_n,a_n}|^2 dx \\
= \;& \frac{-2\alpha_n^{p+1}}{(N-2)^2}\left\{(N-2)^2\omega_N^2 H(a_n,a_n)\lambda_n^{N-2}+O\left(\frac{\lambda_n^N}{d_n^N}|\log(\frac{\lambda_n}{d_n})|\right)\right\} \\
= \;& -2\alpha_n^{p+1}\omega_N^2 H(a_n,a_n)\lambda_n^{N-2}+O\left(\frac{\lambda_n^N}{d_n^N}|\log(\frac{\lambda_n}{d_n})|\right). \qquad (2.30)
\end{aligned}
$$

Here we have used the fact that φ_{λ_n,a_n} is a harmonic function on Ω, $w_n \in E_{\lambda_n,a_n}$ and

$$
\begin{aligned}
&\int_{\mathbf{R}^N}|\nabla\varphi_{\lambda_n,a_n}|^2 dx = \int_{\mathbf{R}^N}|\nabla U_{\lambda_n,a_n}|^2 dx - \int_{\mathbf{R}^N}|\nabla PU_{\lambda_n,a_n}|^2 dx \\
= \;& (N-2)^2\omega_N^2 H(a_n,a_n)\lambda_n^{N-2}+O\left(\frac{\lambda_n^N}{d_n^N}|\log(\frac{\lambda_n}{d_n})|\right) \qquad (2.31)
\end{aligned}
$$

by Lemma 2.1.

Now, we claim that the error term in (2.28) can be estimated as

$$
O\left(\int_{\mathbf{R}^N}U^{p-1}_{\lambda_n,a_n}W_n^2 dx+\int_{\mathbf{R}^N}|W_n|^{p+1}dx\right)=O\left(\|\nabla w_n\|^2_{L^2(\Omega)}\right)+O\left(\frac{\lambda_n^N}{d_n^N}\right). \qquad (2.32)
$$

Indeed, we divide the integral as

$$
\int_{\mathbf{R}^N}U^{p-1}_{\lambda_n,a_n}W_n^2 dx=\int_{\mathbf{R}^N\setminus\Omega}U^{p-1}_{\lambda_n,a_n}W_n^2 dx+\int_{\Omega}U^{p-1}_{\lambda_n,a_n}W_n^2 dx. \qquad (2.33)
$$

Since $W_n = -\alpha_n U_{\lambda_n,a_n}$ on $\mathbf{R}^N\setminus\Omega$, the first term in (2.33) is estimated as

$$
\int_{\mathbf{R}^N\setminus\Omega}U^{p-1}_{\lambda_n,a_n}W_n^2 dx=\alpha_n^2\int_{\mathbf{R}^N\setminus\Omega}U^{p+1}_{\lambda_n,a_n}dx=O\left(\int_{\mathbf{R}^N\setminus B_{d_n}(a_n)}U^{p+1}_{\lambda_n,a_n}dx\right).
$$

Now we compute

$$
\int_{\mathbf{R}^N\setminus B_{d_n}(a_n)}U^{p+1}_{\lambda_n,a_n}dx=\omega_N\int_{d_n}^{\infty}\left(\frac{\lambda_n}{\lambda_n^2+r^2}\right)^N r^{N-1}dr=O(\frac{\lambda_n^N}{d_n^N}),
$$

so we have

$$\int_{\mathbf{R}^N\setminus\Omega} U_{\lambda_n,a_n}^{p-1} W_n^2 dx = O(\frac{\lambda_n^N}{d_n^N}). \tag{2.34}$$

Substituting W_n by $-\alpha_n\varphi_{\lambda_n,a_n}+w_n$ in the second term in (2.33), we have

$$\begin{aligned}\int_\Omega U_{\lambda_n,a_n}^{p-1} W_n^2 dx &= \alpha_n^2 \int_\Omega U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx + \int_\Omega U_{\lambda_n,a_n}^{p-1} w_n^2 dx \\ &+ O\left((\int_\Omega U_{\lambda_n,a_n}^{p-1} w_n^2 dx)^{1/2} (\int_\Omega U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx)^{1/2} \right).\end{aligned} \tag{2.35}$$

Now by Hölder and Sobolev inequality, we find

$$\begin{aligned}\int_\Omega U_{\lambda_n,a_n}^{p-1} w_n^2 dx &= O\left((\int_{\mathbf{R}^N} U_{\lambda_n,a_n}^{p+1} dx)^{\frac{p-1}{p+1}} (\int_\Omega w_n^{p+1} dx)^{\frac{2}{p+1}} \right) \\ &= O(\|\nabla w_n\|_{L^2(\Omega)}^2).\end{aligned} \tag{2.36}$$

On the other hand, when we estimate the first term in (2.35), we divide the integral as

$$\int_\Omega U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx = \int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx + \int_{\Omega\setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx. \tag{2.37}$$

First term in (2.37) is estimated as

$$\begin{aligned}\int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx &= O\left(\|\varphi_{\lambda_n,a_n}\|_{L^\infty(\Omega)}^2 \cdot \int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} dx \right) \\ &= O\left(\left(\frac{\lambda_n^{\frac{N-2}{2}}}{d_n^{N-2}} \right)^2 \cdot \lambda_n^2 d_n^{N-4} \right) = O(\frac{\lambda_n^N}{d_n^N}).\end{aligned} \tag{2.38}$$

Here we have used the fact

$$\int_{B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} dx = \omega_N \int_0^{d_n} \left(\frac{\lambda_n}{\lambda_n^2+r^2} \right)^2 r^{N-1} dr = O(\lambda_n^2 d_n^{N-4}),$$

since $N \geq 5$.
Second term in (2.37) is estimated as before:

$$\int_{\Omega\setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx = O\left(\int_{\mathbf{R}^N\setminus B_{d_n}(a_n)} U_{\lambda_n,a_n}^{p+1} dx \right) = O(\frac{\lambda_n^N}{d_n^N}). \tag{2.39}$$

By (2.37)-(2.39), we have

$$\int_\Omega U_{\lambda_n,a_n}^{p-1} \varphi_{\lambda_n,a_n}^2 dx = O(\frac{\lambda_n^N}{d_n^N}). \tag{2.40}$$

Combining (2.35),(2.36) and (2.40), we obtain

$$\int_\Omega U^{p-1}_{\lambda_n,a_n} W_n^2 dx = O\left(\|\nabla w_n\|^2_{L^2(\Omega)}\right) + O\left(\frac{\lambda_n^N}{d_n^N}\right). \tag{2.41}$$

Finally, by Sobolev inequality and convex inequality $(a+b)^t \le C(a^t + b^t)$ for some $C > 0$ $(a, b > 0, t > 1)$, we have

$$\begin{aligned}
\int_{\mathbf{R}^N} |W_n|^{p+1} dx &= O\left(\left(\int_{\mathbf{R}^N} |\nabla W_n|^2 dx\right)^{\frac{p+1}{2}}\right) \\
&= O\left(\left(\int_{\mathbf{R}^N} |\nabla \varphi_{\lambda_n,a_n}|^2 dx + \int_{\mathbf{R}^N} |\nabla w_n|^2 dx\right)^{\frac{p+1}{2}}\right) \\
&= O\left(\left(\int_{\mathbf{R}^N} |\nabla \varphi_{\lambda_n,a_n}|^2 dx\right)^{\frac{p+1}{2}}\right) + O\left(\left(\int_{\mathbf{R}^N} |\nabla w_n|^2 dx\right)^{\frac{p+1}{2}}\right).
\end{aligned} \tag{2.42}$$

(Recall we extend φ_{λ_n,a_n} to $\mathbf{R}^N \setminus \Omega$ by U_{λ_n,a_n}). So by (2.42), (2.31) and the estimate $H(a_n, a_n) = O(\frac{1}{d_n^{N-2}})$ (see (2.21)), we obtain

$$\begin{aligned}
\int_{\mathbf{R}^N} |W_n|^{p+1} dx &= O\left(\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right)^{\frac{N}{N-2}}\right) + O\left(\|\nabla w_n\|^{\frac{2N}{N-2}}_{L^2(\Omega)}\right) \\
&= O\left(\frac{\lambda_n^N}{d_n^N}\right) + o\left(\|\nabla w_n\|^2_{L^2(\Omega)}\right).
\end{aligned} \tag{2.43}$$

Combining (2.33),(2.34),(2.41) and (2.43), we conclude the claim (2.32).

Returning to (2.28) and using (2.29),(2.30) and (2.32), we obtain

$$S^{N/2} = \alpha_n^{p+1} A - 2\alpha_n^{p+1} \cdot \omega_N^2 H(a_n, a_n)\lambda_n^{N-2} + O\left(\|\nabla w_n\|^2_{L^2(\Omega)}\right) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right).$$

Dividing the both sides by A and noting that $\frac{S^{N/2}}{A} = \alpha_N^{p+1}$, we have

$$\alpha_N^{p+1} = \alpha_n^{p+1} - \alpha_n^{p+1}\left(\frac{2\omega_N^2}{A}\right) H(a_n, a_n)\lambda_n^{N-2} + O\left(\|\nabla w_n\|^2_{L^2(\Omega)}\right) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right).$$

From this we can derive the conclusion. □

Combining Lemma 2.3 and Lemma 2.4, we obtain:

Lemma 2.5.(Asymptotic behavior of S_{ε_n})

As $n \to \infty$,

$$\begin{aligned} S_{\varepsilon_n} &:= \inf_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{L^{p+1}(\Omega)}=1}} \left\{ \int_\Omega |\nabla v|^2 dx - \varepsilon_n \int_\Omega v^2 dx \right\} \\ &= S \cdot S^{-\frac{N}{2}} J_n \\ &= S + S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right) H(a_n, a_n)\lambda_n^{N-2} - \varepsilon_n \left(\frac{S\omega_N C_N}{N(N-2)A}\right)\lambda_n^2 \\ &+ O\left(\|\nabla w_n\|^2_{L^2(\Omega)}\right) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o\left(\varepsilon_n \lambda_n^2\right). \quad (N \geq 5) \end{aligned}$$

$$\begin{aligned} S_{\varepsilon_n} &= S + \frac{S}{2}\left(\frac{\omega_4^2}{A}\right) H(a_n, a_n)\lambda_n^2 - \varepsilon_n \left(\frac{S\omega_4}{8A}\right)\lambda_n^2 |\log \lambda_n| \\ &+ O\left(\|\nabla w_n\|^2_{L^2(\Omega)}\right) + o\left(\frac{\lambda_n^2}{d_n^2}\right) + o\left(\varepsilon_n \lambda_n^2 |\log \lambda_n|\right). \quad (N = 4) \end{aligned}$$

As for the "w-part" of v_n, we have the following estimate due to Rey [13](Appendix C:(C.1)).

Lemma 2.6. *As* $n \to \infty$, *we have*

$$\|\nabla w_n\|^2_{L^2(\Omega)} = o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o(\varepsilon_n \lambda_n^2).$$

Now, we need the appropriate bound of the value S_{ε_n} from the above. The restriction that we consider only least energy solutions is essential in the next lemma.

Lemma 2.7.(Upper bound of S_ε)
For any $a \in \Omega$ *and* $\rho > 0$, *there exists* $\varepsilon_0 = \varepsilon_0(a, \rho)$ *such that if* $\varepsilon \in (0, \varepsilon_0)$, *then the following holds:*

$$\begin{aligned} S_\varepsilon &= \inf_{\substack{v \in H_0^1(\Omega) \\ \|v\|_{L^{p+1}(\Omega)}=1}} \left\{ \int_\Omega |\nabla v|^2 dx - \varepsilon \int_\Omega v^2 dx \right\} \\ &\leq S - \left(\frac{N-4}{N-2}\right)\varepsilon \left\{ \frac{S\omega_N C_N}{N(N-2)A} - \rho \right\} \left[\frac{2C_N \varepsilon}{(N-2)^3 \omega_N H(a,a)} \right]^{\frac{2}{N-4}} \end{aligned}$$

when $N \geq 5$.

$$S_\varepsilon \le S - \frac{S\varepsilon\omega_4}{16Ae}\exp\left(-\frac{8\omega_4 H(a,a)+\varepsilon/e+2\rho}{\varepsilon}\right)$$

when $N=4$.

Proof ($N \ge 5$). For $a \in \Omega$ and $\varepsilon > 0$, define $\psi_{\varepsilon,a} \in H_0^1(\Omega)$ as

$$\psi_{\varepsilon,a} := S^{\frac{-(N-2)}{4}} \alpha_N PU_{\lambda_a(\varepsilon),a}, \tag{2.44}$$

where

$$\lambda_a(\varepsilon) := \left[\frac{2C_N\varepsilon}{(N-2)^3\omega_N H(a,a)}\right]^{\frac{1}{N-4}}. \tag{2.45}$$

Note that $\lambda_a(\varepsilon)$ is the unique minimum point of the function

$$f(\lambda) = K_1 H(a,a)\lambda^{N-2} - K_2\varepsilon\lambda^2 \quad \text{for } \lambda > 0,$$

and it gives the minimum value

$$\min_{\lambda>0} f(\lambda) = f(\lambda_a(\varepsilon)) = -\left(\frac{N-4}{N-2}\right)K_2\varepsilon\left(\frac{2K_2\varepsilon}{(N-2)K_1H(a,a)}\right)^{\frac{2}{N-4}}$$

$$= -\left(\frac{N-4}{N-2}\right)\varepsilon\left(\frac{S\omega_N C_N}{N(N-2)A}\right)\left(\frac{2C_N\varepsilon}{(N-2)^3\omega_N H(a,a)}\right)^{\frac{2}{N-4}}. \tag{2.46}$$

Here, we denote

$$K_1 = S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right), \quad K_2 = \frac{S\omega_N C_N}{N(N-2)A}. \tag{2.47}$$

Define

$$J_\varepsilon(\psi) := \frac{\int_\Omega |\nabla\psi|^2 dx - \varepsilon\int_\Omega \psi^2 dx}{(\int_\Omega |\psi|^{p+1}dx)^{\frac{2}{p+1}}} \tag{2.48}$$

for $\psi \in H_0^1(\Omega)\setminus\{0\}$.

Now, we claim that:

$$\begin{aligned} J_\varepsilon(\psi_{\varepsilon,a}) &= S - \left(\frac{N-4}{N-2}\right)\varepsilon\left\{\frac{S\omega_N C_N}{N(N-2)A}\right\}\left[\frac{2C_N\varepsilon}{(N-2)^3\omega_N H(a,a)}\right]^{\frac{2}{N-4}} \\ &+ o(\varepsilon^{\frac{N-2}{N-4}}) \end{aligned} \tag{2.49}$$

Indeed, as in the calculation in the proof of Lemma 2.1, Lemma 2.2 (note now $d(a, \partial\Omega)$ is a constant independent of ε), we have

$$\begin{aligned}
&\int_\Omega |\nabla \psi_{\varepsilon,a}|^2 dx = S \cdot S^{-\frac{N}{2}} \alpha_N^2 \int_\Omega |\nabla PU_{\lambda_a(\varepsilon),a}|^2 dx \\
&= S - S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right) H(a,a)\lambda_a^{N-2}(\varepsilon) + o(\lambda_a^{N-2}(\varepsilon)), \qquad (2.50) \\
&\int_\Omega \psi_{\varepsilon,a}^2 dx = S \cdot S^{-\frac{N}{2}} \alpha_N^2 \int_\Omega PU_{\lambda_a(\varepsilon),a}^2 dx \\
&= \frac{S\omega_N C_N}{N(N-2)A}\lambda_a^2(\varepsilon) + o(\lambda_a^2(\varepsilon)) \qquad (2.51)
\end{aligned}$$

as $\varepsilon \to 0$.

Also by an argument similar to the one in the proof of Lemma 2.4, we have

$$\begin{aligned}
&\int_\Omega |\psi_{\varepsilon,a}|^{p+1} dx = S^{-\frac{N}{2}} \alpha_N^{p+1} \int_\Omega |PU_{\lambda_a(\varepsilon),a}|^{p+1} dx \\
&= \frac{1}{A}\left\{\int_\Omega U_{\lambda_a(\varepsilon),a}^{p+1} dx + (p+1)\int_\Omega U_{\lambda_a(\varepsilon),a}^{p} \varphi_{\lambda_a(\varepsilon),a} dx \right. \\
&\qquad \left. + O\left(\int_\Omega U_{\lambda_a(\varepsilon),a}^{p-1} \varphi_{\lambda_a(\varepsilon),a}^2 dx + \int_\Omega |\varphi_{\lambda_a(\varepsilon),a}|^{p+1} dx\right)\right\} \\
&= \frac{1}{A}\left\{A - 2\omega_N^2 \lambda_a^{N-2}(\varepsilon) H(a,a) + o(\lambda_a^{N-2}(\varepsilon))\right\} \\
&= 1 - \left(\frac{2\omega_N^2}{A}\right)\lambda_a^{N-2}(\varepsilon) H(a,a) + o(\lambda_a^{N-2}(\varepsilon)). \qquad (2.52)
\end{aligned}$$

Note that $S^{N/2} = \alpha_N^2 N(N-2)A = \alpha_N^{p+1} A$.

So, by (2.50)-(2.52) and $(1+x)^{-\frac{2}{p+1}} = 1 - \frac{2}{p+1}x + o(x)$ as $x \to 0$, we obtain

$$\begin{aligned}
&J_\varepsilon(\psi_{\varepsilon,a}) \\
&= \left\{S - S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right) H(a,a)\lambda_a^{N-2}(\varepsilon) + o(\lambda_a^{N-2}(\varepsilon)) \right. \\
&\qquad \left. - \varepsilon\left(\frac{S\omega_N C_N}{N(N-2)A}\right)\lambda_a^2(\varepsilon) + o(\varepsilon\lambda_a^2(\varepsilon))\right\} \\
&\times \left\{1 + \frac{2}{p+1}\left(\frac{2\omega_N^2}{A}\right) H(a,a)\lambda_a^{N-2}(\varepsilon) + o(\lambda_a^{N-2}(\varepsilon))\right\} \\
&= S + S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right) H(a,a)\lambda_a^{N-2}(\varepsilon) - \varepsilon\left(\frac{S\omega_N C_N}{N(N-2)A}\right)\lambda_a^2(\varepsilon)
\end{aligned}$$

$$
\begin{aligned}
&+ \quad o(\varepsilon\lambda_a^2(\varepsilon)) + o(\lambda_a^{N-2}(\varepsilon)) \\
&= \quad S - \left(\frac{N-4}{N-2}\right)\varepsilon\left\{\frac{S\omega_N C_N}{N(N-2)A}\right\}\left[\frac{2C_N\varepsilon}{(N-2)^3\omega_N H(a,a)}\right]^{\frac{2}{N-4}} \\
&+ \quad o(\varepsilon^{\frac{N-2}{N-4}}) \qquad (2.53)
\end{aligned}
$$

as $\varepsilon \to 0$.

This proves the claim. The last equality in (2.53) follows from our choice of $\lambda_a(\varepsilon)$ (see (2.46)) and the fact

$$\varepsilon\lambda_a^2(\varepsilon) = C_1\lambda_a^{N-2}(\varepsilon) = C_2\varepsilon^{\frac{N-2}{N-4}}$$

by the definition of $\lambda_a(\varepsilon)$ (see (2.45)), where C_1,C_2 are constants independent of ε.

From (2.49) and the definition of S_ε, we obtain the conclusion of Lemma 2.7. □

3. Proof of Theorem. In this section, we prove Theorem 1 by using lemmas we prepared in the previous section.

First, we will show that the blow up point a_∞ is in the interior of Ω.

Indeed, suppose the contrary. Then $a_\infty \in \partial\Omega$ and $d_n = d(a_n, \partial\Omega) \to 0$ as $n \to \infty$. Then by Lemma 2.5, Lemma 2.6 and the estimate (2.21), we can find constants $C_1, C_2, C_3 > 0$ such that

$$
\begin{aligned}
S_{\varepsilon_n} &= S + S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right)H(a_n,a_n)\lambda_n^{N-2} - \varepsilon_n\left(\frac{S\omega_N C_N}{N(N-2)A}\right)\lambda_n^2 \\
&+ O\left(\|\nabla w_n\|^2_{L^2(\Omega)}\right) + o\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) + o\left(\varepsilon_n\lambda_n^2\right) \\
&\geq S + C_1\left(\frac{\lambda_n^{N-2}}{d_n^{N-2}}\right) - C_2\varepsilon_n\lambda_n^2 \\
&\geq S - \left(\frac{N-4}{N-2}\right)C_2\varepsilon_n\left\{\frac{2C_2\varepsilon_n}{(N-2)C_1(\frac{1}{d_n^{N-2}})}\right\}^{\frac{2}{N-4}} \\
&= S - C_3\varepsilon_n^{\frac{N-2}{N-4}}d_n^{\frac{2(N-2)}{N-4}} = S + o(\varepsilon_n^{\frac{N-2}{N-4}}), \qquad (3.1)
\end{aligned}
$$

since we assume $d_n \to 0$.

Here as in the proof of Lemma 2.7, we have used the fact that $f(\lambda) = C_4\lambda^{N-2} - C_5\lambda^2$ has the unique global minimum value $-\left(\frac{N-4}{N-2}\right)C_5\left(\frac{2C_5}{(N-2)C_4}\right)^{\frac{2}{N-4}}$ for $\lambda > 0$, where $C_4 = C_1(\frac{1}{d_n^{N-2}}), C_5 = C_2\varepsilon_n$.

On the other hand, we know that $S_{\varepsilon_n} \le S - C\varepsilon_n^{\frac{N-2}{N-4}} + o(\varepsilon_n^{\frac{N-2}{N-4}})$ for some $C > 0$ (see Lemma 2.7 (2.49)). This contradicts (3.1), so we conclude that a_∞ is in the interior of Ω.

Now, since we have proved that $d_n \ge C$ for some constant $C > 0$ uniformly in n, we may drop d_n in the asymptotic formulas Lemma 2.5 and Lemma 2.6.

Therefore, we can find $p_n > 0, p_n \to 0$ and $q_n > 0, q_n \to 0$ such that

$$\begin{aligned} S_{\varepsilon_n} &= S + S\left(\frac{N-2}{N}\right)\left(\frac{\omega_N^2}{A}\right) H(a_n,a_n)\lambda_n^{N-2} - \varepsilon_n\left(\frac{S\omega_N C_N}{N(N-2)A}\right)\lambda_n^2 \\ &+ o(\lambda_n^{N-2}) + o\left(\varepsilon_n\lambda_n^2\right) \\ &\ge S + (K_1H(a_n,a_n) - p_n)\lambda_n^{N-2} - (K_2+q_n)\varepsilon_n\lambda_n^2 \\ &\ge S - \left(\frac{N-4}{N-2}\right)(K_2+q_n)\varepsilon_n\left[\frac{2(K_2+q_n)\varepsilon_n}{(N-2)(K_1H(a_n,a_n)-p_n)}\right]^{\frac{2}{N-4}} \quad (3.2)\end{aligned}$$

where K_1, K_2 are defined in (2.47). The last inequality of (3.2) follows again by the property of the function $f(\lambda) = C_4\lambda^{N-2} - C_5\lambda^2$.

Combine (3.2) with Lemma 2.7, we have

$$\begin{aligned} S &- \left(\frac{N-4}{N-2}\right)(K_2+q_n)\varepsilon_n\left[\frac{2(K_2+q_n)\varepsilon_n}{(N-2)(K_1H(a_n,a_n)-p_n)}\right]^{\frac{2}{N-4}} \\ &\le S_{\varepsilon_n} \le \\ S &- \left(\frac{N-4}{N-2}\right)(K_2-\rho)\varepsilon_n\left[\frac{2K_2\varepsilon_n}{(N-2)K_1H(a,a)}\right]^{\frac{2}{N-4}}\end{aligned}$$

for any $a \in \Omega$ and $\rho > 0$, if n sufficiently large.

From this we obtain

$$(K_2+q_n)\varepsilon_n\left[\frac{2(K_2+q_n)\varepsilon_n}{(N-2)(K_1H(a_n,a_n)-p_n)}\right]^{\frac{2}{N-4}} \ge (K_2-\rho)\varepsilon_n\left[\frac{2K_2\varepsilon_n}{(N-2)K_1H(a,a)}\right]^{\frac{2}{N-4}}.$$

Dividing both sides by $\varepsilon_n^{\frac{N-2}{N-4}}$ and letting $n \to \infty$, we have

$$K_2\left[\frac{2K_2}{(N-2)K_1H(a_\infty,a_\infty)}\right]^{\frac{2}{N-4}} \ge (K_2-\rho)\left[\frac{2K_2}{(N-2)K_1H(a,a)}\right]^{\frac{2}{N-4}}. \quad (3.3)$$

For $\rho > 0$ can be arbitrary small, (3.3) implies

$$H(a_\infty,a_\infty) \le H(a,a)$$

for any $a \in \Omega$.

Therefore we conclude that a_∞ minimizes the Robin function $H(a,a)$. This completes the proof of Theorem. □

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